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1992 J. Phys. A: Math. Gen. 25 6461

(http://iopscience.iop.org/0305-4470/25/23/038)

J. Phys. A: Math. Gen. 25 (1992) 6461-6469. Printed in the UK

Relativistic analogy of the Aharonov–Bohm effect in the presence of Coulomb field and magnetic charge

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Received 28 February 1992, in final form 9 July 1992

Abstract. We establish the exact solution of the Dirac equation for a charged particle moving in the Coulomb field plus Aharonov-Bohm field plus Dirac monopole field by using the equation in two-dimensional complex space. On the basis of the found solution, the relativistic analogy of the Aharonov–Bohm effect for the system of a Diogen atom existing in an infinite cylindrical solenoid is studied. In this case we obtain some surprising results. One of them is that the Aharonov–Bohm effect is absent for certain states.

1. Introduction

To construct the algebraic method of solving the Dirac equation for the charged particle moving in the Coulomb field, Komarov and Romanova (1985) established the connection between the Dirac equation and the equation for a particle having coordinates of two-component spinors. In this paper we show that the above-mentioned equation in two-dimensional complex space can be used to obtain the exact solution for the Dirac equation for a charged particle moving in the Coulomb field plus Dirac monopole and Aharonov–Bohm fields (section 2). It should be noted that the equation in two-dimensional complex space is more convenient than the Dirac equation for the considered system because of two reasons. First, by transforming from the usual three-dimensional space to the two-dimensional complex space using KS-transformation (see Barut *et al* 1979) the Kepler problem becomes an oscillator problem (see, for example, Komarov and Romanova 1982, Kibler 1983). Second, the 'extra' variable in the KS-transformation can be used to describe the Dirac monopole field (Iwai *et al* 1986). Thus, for the equation written in two-dimensional complex space, the Dirac monopole potential can be introduced without an overt form. Second, for the above-mentioned equation we can use the parameters of group SU(2) (Fedorov 1979) to make the separation of variables in the equation (section 3). In section 4 we will establish the exact solution of the considered equation. The angular function can be obtained simply by using the algebraic correlations between operators of parameters of group SU(2). The radial equation of the system is standard and well known. The Aharonov–Bohm effect is of great interest to many researchers (see, for example, Afanasiev 1990). Considering this effect, the majority of investigators deal with the case of scattering of an electron beam by an infinite solenoid. However, there is another way to look at this problem. It is to study the spectrum of an atom, putting it in the field of an infinite solenoid. Actually, it is the problem of an electron, moving in the Coulomb field plus the Aharonov–Bohm field. In our case, to these fields we add the

Dirac monopole field. The exact solution found will enable us to make some conclusions about the relativistic analogy of the Aharonov–Bohm effect with the presence of the Coulomb field and a magnetic charge. One of them is that the Aharonov–Bohm effect is absent for certain quantum states of the particle. Second, if the magnetic flux inside the solenoid (the source of the Aharonov–Bohm field) changes slowly (adiabatically), then the conservation of the starting quantum state only continues until a certain critical value of the magnetic flux. Further change in this value leads to an inevitable sudden leap of the electron state.

2. The connection between the equations

In this section we will establish the connection between the equation in two-dimensional complex space and the Dirac equation for the charged particle moving in the Coulomb field plus the Dirac monopole field plus the Aharonov–Bohm field, and show the relationship between the eigenfunctions of these equations.

Let us consider the equation (see Komarov and Romanova 1985)

$$\mathbf{H}\Psi(\xi) = Ze^2\Psi(\xi) \tag{1}$$

$$\mathbf{H} = -\tfrac{1}{2}\mathrm{i}\hbar c\alpha_\lambda(\tau_\lambda)_{st}\left(\xi_t \frac{\partial}{\partial\xi_s} + \xi_s^* \frac{\partial}{\partial\xi_t^*}\right) + (e\alpha_\lambda A_\lambda + mc^2\beta - \varepsilon)\xi_s\xi_s^* \tag{2}$$

where the four-component spinor $\Psi(\xi)$ is a function of the complex coordinates ξ_s ($s=1,2$); α_λ ($\lambda=1,2,3$) and β are Dirac matrices. Below in this paper we use the usual representation

$$\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} \qquad \alpha_\lambda = \begin{pmatrix}0 & \sigma_\lambda\\ \sigma_\lambda & 0\end{pmatrix} \qquad \beta = \begin{pmatrix}0 & 1\\ 0 & -1\end{pmatrix} \tag{3}$$

where Ψ_1 and Ψ_2 are two-component spinors, and σ_λ ($\lambda=1,2,3$) are Pauli matrices. In the Hamiltonian (2) $(\tau_\lambda)_{st}$ are the matrix elements of Pauli matrices, operating in the space of coordinates ξ, which are regarded as spinor components; the asterisk denotes the complex conjugate operation. We regard the function A_λ as being invariant with respect to the transformation (α = arbitrary real constant):

$$\xi_s \to \mathrm{e}^{\mathrm{i}\alpha}\xi_s \qquad \xi_s^* \to \mathrm{e}^{-\mathrm{i}\alpha}\xi_s^* \qquad \frac{\partial}{\partial\xi_s} \to \mathrm{e}^{-\mathrm{i}\alpha}\frac{\partial}{\partial\xi_s} \qquad \frac{\partial}{\partial\xi_s^*} \to \mathrm{e}^{\mathrm{i}\alpha}\frac{\partial}{\partial\xi_s^*}$$

that follows the invariance of the Hamiltonian (2) by the above written transformation. Hence we have

$$\mathbf{HQ} - \mathbf{QH} = 0$$

where

$$\mathbf{Q} = \xi_s^*\frac{\partial}{\partial\xi_s^*} - \xi_s\frac{\partial}{\partial\xi_s}. \tag{4}$$

Let us now make the substitution of variables in equation (1) using the following correlations:

$$\begin{aligned} x_\lambda &= \xi_s^*(\tau_\lambda)_{st}\xi_t \qquad \lambda = 1,2,3\\ f &= \tan^{-1}(\operatorname{Im}\xi_1/\operatorname{Re}\xi_1) \qquad (0 \leqslant f \leqslant 2\pi). \end{aligned} \tag{5}$$

In view of the assumed properties of ξ_s, the variables x_λ ($\lambda = 1, 2, 3$) constitute the components of three-dimensional real vector $\boldsymbol{r}$ and $r = \sqrt{x_\lambda x_\lambda} = \xi_s \xi_s^*$. In the new variables the operators $\mathbf{Q}$ and $\mathbf{H}$ take the form

$$\mathbf{Q} = \mathrm{i}\frac{\partial}{\partial f} \tag{6}$$

$$\mathbf{H} = -\mathrm{i}\hbar c r \alpha_\lambda \frac{\partial}{\partial x_\lambda} - \hbar c \frac{x_2}{2(r+x_3)} \mathbf{Q}\alpha_1 + \hbar c \frac{x_1}{2(r+x_3)} \mathbf{Q}\alpha_2 + r(e\alpha A + mc^2\beta - \varepsilon) \tag{7}$$

where the functions A_λ depend on the variables x_λ only. From (7) we can easily show that the variables $\boldsymbol{r}$ and f in equation (1) are separated, and that, as follows from (6), we write

$$\Psi(\boldsymbol{r}, f) = \exp(2\mathrm{i}qf)\psi(\boldsymbol{r}). \tag{8}$$

Then

$$\mathbf{Q}\Psi = -2p\Psi \tag{9}$$

and from equation (1) we obtain the following equation:

$$\left\{-\mathrm{i}\hbar c \alpha_\lambda \frac{\partial}{\partial x_\lambda} + \hbar c q \frac{1}{r(r+x_3)}(x_2\alpha_1 - x_1\alpha_2) + e\alpha_\lambda A_\lambda(\boldsymbol{r}) + mc^2\beta - \frac{Ze^2}{r} - \varepsilon\right\}\psi(\boldsymbol{r}) = 0. \tag{10}$$

Taking into account that $\Psi(r, f)$ is a single-valued function, the quantum number q can be expressed as

$$q = 0, \pm\tfrac{1}{2}, \pm 1, \pm\tfrac{3}{2}, \ldots. \tag{11}$$

In this view we see that: (i) if $q = 0$ equation (10) is the Dirac equation for a particle with mass m and charge $-e$, moving in the Coulomb field Ze/r plus the field which is represented by the potential vector $A_\lambda(\boldsymbol{r})$; (ii) when $q \neq 0$ then to the mentioned fields is added the field of the Dirac monopole, situated in the origin of the coordinates system, with the magnetic charge

$$g = \frac{\hbar c}{e} q$$

satisfying the Dirac quantum condition (see Dirac 1979)†.

From the above, it follows that the solution of the Dirac equation for a particle moving in our specified fields can be found by using equation (1). The wavefunctions in the usual three-dimensional space can be obtained from the solution of equation (1) making the substitution of variables (5) and then writing Im $\xi_1 = 0$.

3. The separation of the variables in equation (1)

As it was shown in section 2, to get exact solutions to the Dirac equation for the considered system, we can use equations (1) and (2) in which the potential vector A_λ

† The possibility of monopole description by the use of the 'extra' coordinate has been used in many papers (see for example Iwai *et al* 1986).

has the following components:

$$A_1=-\frac{F}{2\pi}\frac{\mathrm{i}(\xi_2^*\xi_1-\xi_1^*\xi_2)}{4\xi_1^*\xi_1\xi_2^*\xi_2}$$
$$A_2=\frac{F}{2\pi}\frac{\xi_1^*\xi_2+\xi_2^*\xi_1}{4\xi_1^*\xi_1\xi_2^*\xi_2} \tag{12}$$
$$A_3=0 \qquad (F=\text{constant}).$$

The potential vector (12) corresponds to the Aharonov–Bohm potential in the usual coordinates $\boldsymbol{r}$

$$A_1=-\frac{F}{2\pi}\frac{x_2}{x_1^2+x_2^2} \qquad A_2=\frac{F}{2\pi}\frac{x_1}{x_1^2+x_2^2} \qquad A_3=0$$

(see Aharonov and Bohm 1959). To separate the variables in equation (1), it is more suitable to use the new real variables r and a_λ ($\lambda=1,2,3$), which are introduced as

$$\xi_s=\sqrt{r}\left(\frac{1-\mathrm{i}a_\lambda\tau_\lambda}{\sqrt{1+\boldsymbol{a}^2}}\right)_{st}\eta_t \qquad \xi_s^*=\sqrt{r}\,\eta_t^*\left(\frac{1+\mathrm{i}a_\lambda\tau_\lambda}{\sqrt{1+\boldsymbol{a}^2}}\right)_{ts} \tag{13}$$

where $\boldsymbol{a}^2=a_{\lambda_s}a_{\lambda_s}$; η_{0s} ($s=1,2$) are components of a constant spinor, satisfying the condition $\eta_s\eta_s^*=1$ (we will regard $\eta_s=\delta_{s1}$). One can see from (13) that the matrix

$$\frac{1-\mathrm{i}a_\lambda\tau_\lambda}{\sqrt{1+\boldsymbol{a}^2}}$$

is unitary and unimodular, and thereby, owing to the above-mentioned property of coordinates ξ_s, is a representation of the group of the transformation in two-dimensional complex space, which keeps invariant the form $\xi_s\xi_s^*$. It means that the variables a_λ ($\lambda=1,2,3$) represent parameters of the group SU(2). Taking into account the local isomorphism between the group SU(2) and the group of rotation in three-dimensional space SO(3), the parameters a_λ ($\lambda=1,2,3$) have the following physical meanings:

(i) the direction of the vector $\boldsymbol{a}$ coincides with the direction of the rotation axis;

(ii) $|\boldsymbol{a}|=|\tan(\varphi/2)|$, where φ is an angle of rotation around this axis (see Fedorov 1979).

From the definition (13) it follows that

$$r=\xi_s^*\xi_s \qquad x_\lambda=\xi_s^*(\tau_\lambda)_{st}\xi_t=rO_{\lambda\mu}(\boldsymbol{a})n_\mu$$
$$\frac{\partial}{\partial\xi_s}=\sqrt{r}\,\eta_u^*\left(\frac{1+\mathrm{i}a_\lambda\tau_\lambda}{\sqrt{1+\boldsymbol{a}^2}}\right)_{us}\frac{\partial}{\partial r}-\frac{1}{\sqrt{r}}\eta_u^*\left(\frac{1+\mathrm{i}a_\lambda\tau_\lambda}{\sqrt{1+\boldsymbol{a}^2}}\tau_\mu\right)_{us}\mathsf{l}_\mu(\boldsymbol{a}) \tag{14}$$
$$\frac{\partial}{\partial\xi_s^*}=\sqrt{r}\left(\frac{1-\mathrm{i}a_\lambda\tau_\lambda}{\sqrt{1+\boldsymbol{a}^2}}\right)_{su}\eta_u\frac{\partial}{\partial r}+\frac{1}{\sqrt{r}}\left(\tau_\mu\frac{1-\mathrm{i}a_\lambda\tau_\lambda}{\sqrt{1+\boldsymbol{a}^2}}\right)_{su}\eta_u\mathsf{l}_\mu(\boldsymbol{a})$$

where

$$n_\mu=\eta_s^*\tau_\mu)_{st}\eta_t \qquad n_\mu n_\mu=1$$
$$\mathsf{l}_\lambda(\boldsymbol{a})=-\frac{1}{2}\mathrm{i}\left(\frac{\partial}{\partial a_\lambda}+a_\lambda a_\mu\frac{\partial}{\partial a_\mu}+\varepsilon_{\lambda\mu\upsilon}a_\mu\frac{\partial}{\partial a_\upsilon}\right)$$
$$O_{\lambda\mu}(\boldsymbol{a})=\delta_{\lambda\mu}+\frac{2}{1+a^2}(a_\lambda a_\mu-a^2\delta_{\lambda\mu}-\varepsilon_{\lambda\mu\upsilon}a_\upsilon) \tag{15}$$
$$O_{\lambda\upsilon}(\boldsymbol{a})O_{\mu\upsilon}(\boldsymbol{a})=O_{\upsilon\lambda}(\boldsymbol{a})O_{\upsilon\mu}(\boldsymbol{a})=\delta_{\lambda\mu}$$

($\delta_{\lambda\mu}$ and $\varepsilon_{\lambda\mu\nu}$ are Kroneker delta and Levi-Civita symbols respectively). For further use we write the correlations of operators (15) as

$$\begin{aligned}
&[\mathbf{l}_\lambda(\boldsymbol{a}), \mathbf{l}_\mu(\boldsymbol{a})] = \mathrm{i}\varepsilon_{\lambda\mu\upsilon}\mathbf{l}_\upsilon(\boldsymbol{a}) \qquad [\mathbf{l}_\lambda(\boldsymbol{a}), \mathbf{l}_\mu(-\boldsymbol{a})] = 0\\
&[\mathbf{l}_\lambda(\boldsymbol{a}), O_{\mu\upsilon}(\boldsymbol{a})] = \mathrm{i}\varepsilon_{\lambda\mu\rho}O_{\rho\upsilon}(\boldsymbol{a}) \qquad [\mathbf{l}_\lambda(-\boldsymbol{a}), O_{\mu\upsilon}(\boldsymbol{a})] = \mathrm{i}\varepsilon_{\lambda\upsilon\rho}O_{\mu\rho}(\boldsymbol{a})\\
&O_{\lambda\mu}(\boldsymbol{a})\mathbf{l}_\lambda(\boldsymbol{a}) = -\mathbf{l}_\mu(-\boldsymbol{a}) \qquad \varepsilon_{\lambda\mu\upsilon}O_{\lambda\alpha}(\boldsymbol{a})O_{\mu\beta}(\boldsymbol{a}) = \varepsilon_{\alpha\beta\gamma}O_{\upsilon\gamma}(\boldsymbol{a})\\
&\varepsilon_{\lambda\mu\upsilon}O_{\alpha\lambda}(\boldsymbol{a})O_{\beta\mu}(\boldsymbol{a}) = \varepsilon_{\alpha\beta\gamma}O_{\gamma\upsilon}(\boldsymbol{a}).
\end{aligned} \tag{16}$$

Considering equation (1) as a starting equation, it is evident that the scalar product of the wavefunction in the ξ-space is defined as

$$\langle\Psi|\Phi\rangle = \int \mathrm{d}\xi_1' \int \mathrm{d}\xi_1'' \int \mathrm{d}\xi_2' \int \mathrm{d}\xi_2''\, \Psi^+(\xi_1', \xi_1'', \xi_2', \xi_2'')\Phi(\xi_1', \xi_1'', \xi_2', \xi_2'') \tag{17}$$

where $\xi_s' = \mathrm{Re}\,\xi_s$ and $\xi_s'' = \mathrm{Im}\,\xi_s$. The substitution of variables (13) leads (17) to the formula

$$\langle\Psi|\Phi\rangle = \frac{1}{2}\int_0^\infty \mathrm{d}r\, r \int \frac{\mathrm{d}a_1\,\mathrm{d}a_2\,\mathrm{d}a_3}{(1+\boldsymbol{a}^2)^2}\Psi^+(r, \boldsymbol{a})\Phi(r, \boldsymbol{a}). \tag{18}$$

The operators $\mathbf{l}_\lambda(\boldsymbol{a})$ ($\lambda = 1, 2, 3$) are Hermitian with respect to scalar product (18) and represent the infinitesimal transformation operators of group SU(2) (see Fedorov 1979). Later on, we shall see that all calculations can be made algebraically by using formulae (16), which certainly do not change with the other choice of parameters.

Replacing (12)-(14) in (1) and (2), using the representation (3) and taking into account that the Aharonov–Bohm field can be excluded with the aid of the substitution

$$\Psi_{1,2} = \exp\left\{-\mathrm{i}\frac{eF}{2\pi\hbar c}\tan^{-1}\frac{O_{23}(\boldsymbol{a})}{O_{13}(\boldsymbol{a})}\right\}\tilde{\Psi}_{1,2} \tag{19}$$

we lead (1) and (2) to the equations

$$\begin{aligned}
&-\mathrm{i}\hbar c\left\{\sigma_\lambda O_{\lambda 3}(\boldsymbol{a})\left(r\frac{\partial}{\partial r}+1\right) - [\mathbf{l}_3(-\boldsymbol{a}) + \sigma_\lambda O_{\lambda 3}(\boldsymbol{a})(1+\sigma_\mu \mathbf{l}_\mu(\boldsymbol{a}))]\right\}\tilde{\Psi}_2\\
&\qquad + ((mc^2-\varepsilon)r - Ze^2)\tilde{\Psi}_1 = 0\\
&-\mathrm{i}\hbar c\left\{\sigma_\lambda O_{\lambda 3}(\boldsymbol{a})\left(r\frac{\partial}{\partial r}+1\right) - [\mathbf{l}_3(-\boldsymbol{a}) + \sigma_\lambda O_{\lambda 3}(\boldsymbol{a})(1+\sigma_\mu \mathbf{l}_\mu(\boldsymbol{a}))]\right\}\tilde{\Psi}_1\\
&\qquad - ((mc^2+\varepsilon)r + Ze^2)\tilde{\Psi}_2 = 0.
\end{aligned} \tag{20}$$

The operator $\mathbf{\Omega}$ (see Komarov *et al* 1985), by the variables r and $\boldsymbol{a}$, acquires the form

$$\mathbf{\Omega} = -2n_\alpha \mathbf{l}_\alpha(-\boldsymbol{a}) = -2\mathbf{l}_3(-\boldsymbol{a}). \tag{21}$$

The suitable form of equation (20) can be obtained by using the transformation (later, we will consider only the bound states of a Dirac particle, i.e. we will regard $\varepsilon < mc^2$):

$$\tilde{\Psi}_1 = \sqrt{mc^2+\varepsilon}\,(\Phi_1+\Phi_2) \qquad \tilde{\Psi}_2 = \mathrm{i}\sqrt{mc^2-\varepsilon}\,\sigma_\lambda O_{\lambda 3}(\boldsymbol{a})(\Phi_1-\Phi_2). \tag{22}$$

Denoting

$$\hbar\omega = \sqrt{m^2c^4-\varepsilon^2} \qquad \hat{\chi} = \sigma_\lambda O_{\lambda 3}(\boldsymbol{a})\mathbf{l}_3(-\boldsymbol{a}) + \sigma_\lambda \mathbf{l}_\lambda(\boldsymbol{a}) + 1 \tag{23}$$

and putting (22) into (20) we obtain the following equations

$$\begin{aligned}&\left(r\frac{\partial}{\partial r}+1+\frac{\omega}{c}r-\frac{Ze^2\varepsilon}{\hbar^2 c\omega}\right)\Phi_1-\left(\hat{\chi}+\frac{Ze^2mc}{\hbar^2\omega}\right)\Phi_2=0\\ -&\left(\hat{\chi}-\frac{Ze^2mc}{\hbar^2\omega}\right)\Phi_1+\left(r\frac{\partial}{\partial r}+1-\frac{\omega}{c}r+\frac{Ze^2\varepsilon}{\hbar^2 c\omega}\right)\Phi_2=0.\end{aligned}\tag{24}$$

As follows from (24) the variables r and $\boldsymbol{a}$ can be separated if we write

$$\Phi_{1,2}(r,\boldsymbol{a})=\mathbf{F}_{1,2}(r)\mathbf{D}_{pq}^{J\lambda}(\boldsymbol{a})\tag{25}$$

where the functions $\mathbf{D}_{pq}^{J\chi}(\boldsymbol{a})$ are the eigenfunctions of the operator $\hat{\chi}$, i.e.

$$\hat{\chi}\mathbf{D}_{pq}^{J\chi}(\boldsymbol{a})=\chi\mathbf{D}_{pq}^{J\chi}\boldsymbol{a}).\tag{26}$$

With the use of formula (16) it is easy to see that the operators $\mathbf{Q}$ (i.e. $\mathbf{l}_3(-\boldsymbol{a})$), $\mathbf{J}_\lambda=\mathbf{l}_\lambda(\boldsymbol{a})+\frac{1}{2}\sigma_\lambda$ commute with the operator $\hat{\chi}$, moreover $[\mathbf{Q},\mathbf{J}_\lambda]=0$. Therefore, we will define the functions $\mathbf{D}_{pq}^{J\chi}(\boldsymbol{a})$ by equation (26) and by the following equations

$$\begin{aligned}&\mathbf{J}_\lambda(\boldsymbol{a})\mathbf{J}_\lambda(\boldsymbol{a})\mathbf{D}_{pq}^{J\chi}\boldsymbol{a})=J(J+1)\mathbf{D}_{pq}^{J\chi}(\boldsymbol{a})\\ &J_3(\boldsymbol{a})\mathbf{D}_{pq}^{J\chi}(\boldsymbol{a})=(p+\tfrac{1}{2})\mathbf{D}_{pq}^{J\chi}(\boldsymbol{a})\\ &\mathbf{l}_3(-\boldsymbol{a})\mathbf{D}_{pq}^{J\chi}(\boldsymbol{a})=q\mathbf{D}_{pq}^{J\chi}(\boldsymbol{a})\end{aligned}\tag{27}$$

that stipulate the choice of the designation for functions $\mathbf{D}_{pq}^{J\chi}(\boldsymbol{a})$.

We can make another remark about equation (24). Excluding one of the functions (Φ_1 and Φ_2) from (24) we obtain the equations

$$\left\{\left(r\frac{\partial}{\partial r}+1+\frac{\omega}{c}r-\frac{Ze^2\varepsilon}{\hbar^2c\omega}\right)\left(r\frac{\partial}{\partial r}+1-\frac{\omega}{c}r+\frac{Ze^2\varepsilon}{\hbar^2c\omega}\right)-\left(\hat{\chi}^2-\frac{Z^2e^4m^2c^2}{\hbar^4\omega^2}\right)\right\}\Phi_2=0\tag{28a}$$

$$\left\{\left(r\frac{\partial}{\partial r}+1-\frac{\omega}{c}r+\frac{Ze^2\varepsilon}{\hbar^2c\omega}\right)\left(r\frac{\partial}{\partial r}+1+\frac{\omega}{c}r-\frac{Ze^2\varepsilon}{\hbar^2c\omega}\right)-\left(\hat{\chi}^2-\frac{Z^2e^4m^2c^2}{\hbar^4\omega^2}\right)\right\}\Phi_1=0\tag{28b}$$

from which follows the existence of supersymmetry in the considered problem. Indeed, the operator $\sigma_\lambda O_\lambda(\boldsymbol{a})$, anti-commuting with the operator $\hat{\chi}$, commutes with all of the operators included in (28a), (28b) and (27). Consequently, this circumstance leads to the typical, for super-symmetrical Hamiltonian, doubling of quantum states with the same values of energy and the quantum numbers J, p, q.

4. Exact solution to equation (1)

For the determination of the angular dependence of the wavefunction we start by solving the equations for the generalized spherical functions $\mathbf{E}_{pq}^L(\boldsymbol{a})$:

$$\begin{aligned}&\mathbf{l}_\lambda(\boldsymbol{a})\mathbf{l}_\lambda(\boldsymbol{a})\mathbf{E}_{pq}^L(\boldsymbol{a})=L(L+1)\mathbf{E}_{pq}^L(\boldsymbol{a})\\ &\mathbf{l}_3(\boldsymbol{a})E_{pq}^L(\boldsymbol{a})=p\mathbf{E}_{pq}^L(\boldsymbol{a})\\ &\mathbf{l}_a(-\boldsymbol{a})\mathbf{E}_{pq}^L(\boldsymbol{a})=q\mathbf{E}_{pq}^L(\boldsymbol{a}).\end{aligned}\tag{29}$$

With the use of correlations (16) it is easy to make sure that the solutions of equations (29) can be written in the form

$$\mathbf{E}_{pq}^L(\boldsymbol{a})=\exp\left\{ip\tan^{-1}\frac{O_{23}(\boldsymbol{a})}{O_{13}(\boldsymbol{a})}+iq\tan^{-1}\frac{O_{32}(\boldsymbol{a})}{O_{31}(\boldsymbol{a})}\right\}\mathbf{G}_{pq}^L(O_{33}(\boldsymbol{a})).\tag{30}$$

The allowed values of q are found from the single-valued condition of the wavefunction. The determination in two-dimensional complex space

$$\xi_1=\sqrt{r}\,e^{if}\cos\frac{\vartheta}{2}\qquad \xi_2=\sqrt{r}\,e^{i(f+\varphi)}\sin\frac{\vartheta}{2} \tag{31}$$

corresponds to the spherical coordinates r, ϑ, φ in the usual three-dimensional space. With the use of (13) we find that in these coordinates

$$\mathbf{E}^L_{pq}(\vartheta,\varphi,f)=e^{i(p+q)\varphi+2iqf}\mathbf{G}^L_{pq}(\cos\vartheta). \tag{32}$$

If the previous substitution (19) is taken into account, then from (32) it follows that

$$p+q-\frac{eF}{2\pi\hbar c}=\mathbf{M}\qquad \mathbf{M}=0,\pm1,\pm2,\pm3,\ldots \tag{33}$$

and for the value q we have the former result (11).

The functions $\mathbf{G}^L_{pq}(x)$ $(x=O_{33}(\boldsymbol{a}))$ are defined by the first of equations (29) with the boundary conditions, deriving from the assumption that the source of the Aharonov–Bohm field (12) is an infinitely long cylindrical solenoid (inaccessible for an electron), whose axis is along x_3, and radius $R\to0$. It means that the function $\mathbf{G}^L_{pq}(x)$ must vanish if $x=\pm1$. Putting (30) into the first of equations (29), we find that this equation becomes the well-studied equation for the hypergeometric functions and its solutions can be represented in the form

$$\mathbf{G}^L_{pq}(x)=\begin{cases} e^{i\pi p}\tilde{\mathbf{G}}^L_{pq}(x) & \text{for } p+q>0\\ \tilde{\mathbf{G}}^L_{pq}(x) & \text{for } p+q\leqslant 0\end{cases} \tag{34}$$

where

$$\tilde{\mathbf{G}}^L_{pq}(x)=\left(\frac{\mathscr{L}!(2\mathscr{L}+2s+2t+1)\Gamma(\mathscr{L}+2s+2t+1)}{2^{2s+2t+1}\pi\Gamma(\mathscr{L}+2s+1)\Gamma(\mathscr{L}+2t+1)}\right)^{1/2}(1-x)^s(1+x)^t\mathbf{P}^{(2s,2t)}_L(x) \tag{35}$$

$$s=\tfrac{1}{2}|p+q|\qquad t=\tfrac{1}{2}|p-q|\qquad L=\mathscr{L}+s+t\qquad \mathscr{L}=0,1,2,\ldots.$$

Then we have

$$s=\tfrac12(p+q)>0\qquad t=\tfrac12(p-q)>0\qquad L=\mathscr{L}+p \tag{36a}$$

$$s=\tfrac12(p+q)>0\qquad t=\tfrac12(q-p)>0\qquad L=\mathscr{L}+q \tag{36b}$$

$$s=-\tfrac12(p+q)>0\qquad t=\tfrac12(p-q)>0\qquad L=\mathscr{L}-q \tag{36c}$$

$$s=-\tfrac12(p+q)>0\qquad t=\tfrac12(q-p)>0\qquad L=\mathscr{L}-p. \tag{36d}$$

In the formula (35) $\Gamma(Z)$ is the Euler γ-function and $\mathbf{P}^{(a,b)}_N(x)$ are the Jacobi polynomials (see, for example, Korn *et al* (1989).

With the use of the function $\mathbf{E}^L_{pq}(\boldsymbol{a})$ we can, in a simple way, build solutions of equations (26) and (27):

$$\mathbf{D}^{J\chi_\pm}_{pq}(\boldsymbol{a})=\left(\frac12+\frac{\chi_\pm}{2J+1}\right)^{1/2}\begin{bmatrix}\left(\dfrac{J+\frac12+p}{2J}\right)^{1/2}\mathbf{E}^{J-1/2}_{pq}(\boldsymbol{a})\\ -\left(\dfrac{J-\frac12-p}{2J}\right)^{1/2}\mathbf{E}^{J-1/2}_{p+1,q}(\boldsymbol{a})\end{bmatrix} \pm\left(\frac12-\frac{\chi_\pm}{2J+1}\right)^{1/2}\begin{bmatrix}-\left(\dfrac{J+\frac12-p}{2J+2}\right)^{1/2}\mathbf{E}^{J+1/2}_{pq}(\boldsymbol{a})\\ \left(\dfrac{J+\frac32+p}{2J+2}\right)^{1/2}\mathbf{E}^{J+1/2}_{p+1,q}(\boldsymbol{a})\end{bmatrix} \tag{37}$$

where

$$\chi_{\pm} = \pm\sqrt{(J+\tfrac{1}{2})^2 - q^2}. \tag{38}$$

Putting (25) into equations (24), we obtain the equations for the radial function, which really coincide with the equations for the radial function of the Dirac particle moving in the Coulomb field, the only difference being the set of eigenvalues of the operator $\hat{\chi}$. Therefore, their solution, which belongs to the eigenvalue of the energy

$$\varepsilon = mc^2\left(1 + \frac{Z^2e^4}{\hbar^2c^2[N + (\chi^2 - Z^2e^4/\hbar^2c^2)^{1/2}]^2}\right)^{-1/2} \qquad N = 0, 1, 2, \ldots \tag{39}$$

can be found in any book on relativistic quantum mechanics.

5. Discussion of the found solution

The energy spectrum (39) and functions (37) should now be analysed. From (33)–(39) one can see that the quantum states of the considered system are set by numbers N, $\mathscr{L}$, M; moreover

$$N = 0, 1, 2, \ldots \qquad \mathscr{L} = 0, 1, 2, \ldots \qquad M = 0, \pm 1, \pm 2, \ldots.$$

Putting into (36), concrete values for the quantum numbers corresponding to the wavefunctions $\mathbf{E}^L_{pq}(x)$ in (37), we find

$$J + \tfrac{1}{2} = \mathscr{L} + p \qquad \text{for } |q| \leqslant p \leqslant J - \tfrac{3}{2} \tag{40a}$$

$$J + \tfrac{1}{2} = \mathscr{L} + |q| + 1 \qquad \text{for } -|q| \leqslant p \leqslant |q| - 1 \tag{40b}$$

$$J + \tfrac{1}{2} = \mathscr{L} - p + 1 \qquad \text{for } -J + \tfrac{1}{2} \leqslant p \leqslant -|q| - 1 \tag{40c}$$

where $p = M + eF/2\pi\hbar c - q$.

In the value range of the quantum number p in (40), the values satisfying

$$|q| - 1 < p < |q| \qquad \text{and} \qquad -|q| - 1 < p < -|q| \tag{41}$$

are absent. This means that values do not satisfy the condition (36) that one of the functions $\mathbf{E}^L_{pq}(x)$ in (37) does not vanish when $x = \pm 1$. Thus by fixing the value of the magnetic flux $(F/2\pi)$ in the solenoid, there are some quantum states in which a particle cannot exist. Namely, the states, given by the value M, in the following value ranges:

$$\begin{gathered} q + |q| - 1 - \frac{eF}{2\pi\hbar c} < M < q + |q| - \frac{eF}{2\pi\hbar c} \\ q - |q| - 1 - \frac{eF}{2\pi\hbar c} < M < q - |q| - \frac{eF}{2\pi\hbar c} \end{gathered} \tag{42}$$

are forbidden for the particle moving in the Coulomb field plus Aharonov–Bohm Dirac monopole fields. Specifically, for the states ($\mathscr{L} = 0$) the restriction of the number M is given from (42) and from the 'non fall to the centre' condition

$$\chi^2 - \frac{Z^2e^4}{\hbar^2c^2} > 0. \tag{43}$$

From correlations (40) another interesting result follows. This is the energy spectrum dependence, of the considered system, on the magnetic flux $(F/2\pi)$ in the solenoid (39), i.e. how the Aharonov-Bohm goes on this case. From (40) and (33) we have

$$J+\tfrac{1}{2}=\mathscr{L}+M-q+\frac{eF}{2\pi\hbar c} \qquad \text{for } q+|q|-M\leqslant\frac{eF}{2\pi\hbar c} \tag{44a}$$

$$J+\tfrac{1}{2}=\mathscr{L}+|q|+1 \qquad \text{for } q-|q|-M\leqslant\frac{eF}{2\pi\hbar c}\leqslant q+|q|-M-1 \tag{44b}$$

$$J+\tfrac{1}{2}=\mathscr{L}+1-M+q-\frac{eF}{2\pi\hbar c} \qquad \text{for } \frac{eF}{2\pi\hbar c}\leqslant q-|q|-M-1. \tag{44c}$$

Putting (44) into (38) and (39) one can see that the energy ε of the particle, existing in the defined quantum state, is independent of the magnetic flux when the magnetic flux has a value in the range (44*b*), i.e. in this case the Aharonov-Bohm effect is absent.

Moreover, from equations (44) it follows that the considered system has no quantum state $(N, \mathscr{L}, M)$ for the values of the magnetic flux in the ranges

$$q+|q|-M-1<\frac{eF}{2\pi\hbar c}<q+|q|-M$$

and

$$q-|q|-M-1<\frac{eF}{2\pi\hbar c}<q-|q|-M.$$

This means that if the magnetic flux changes slowly (adiabatically), the conservation of the starting quantum state $(N, \mathscr{L}, M)$ only continues until a certain critical value of the magnetic flux. A further change in the value $F/2\pi$ leads to an inevitable sudden leap of the state of the system, i.e. to a breach of the adiabatical principle.

References

Afanasiev G N 1990 *Phys. Elem. Part. Nucl. Atom.* **21** 172–250
Aharonov Y and Bohm D 1959 *Phys. Rev.* **115** 485–91
Barut A O, Schneider C K E and Wilson R 1979 *J. Math. Phys.* **20** 2244–56
Dirac P 1979 *Principles of Quantum Mechanics* (Moscow: Nauka)
Fedorov F I 1979 *The Lorentz Group* (Moscow: Nauka)
Iwai T and Uwano Y 1986 *J. Math. Phys.* **27** 1523–9
Kibler M and Negadi T 1983 *J. Phys. A: Math. Gen.* **16** 4265–8
Komarov L I and Romanova T S 1982 *Izv. Acad. Nauk BSSR* **2** 98–103
—— 1985 *J. Phys. A: Math. Gen.* **18** 859–65
Korn G and Korn T 1989 *Reference Book of Mathematics* (Moscow: Nauka)